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Accelerometer-based Tilt Estimation of a Rigid Body with only Rotational Degrees of Freedom

Sebastian Trimpe and Raffaello D'Andrea

Abstract— **An estimation algorithm is developed for determining pitch and roll angles (tilt) of a rigid body fixed at a pivot point using multiple accelerometers. The estimate is independent of the rigid body dynamics; the method is applicable both in static conditions and for any dynamic motion of the body. No dynamic model is required for the estimator; only the mounting positions of the sensors need to be known. The proposed estimator is the optimal linear estimate in a least-squares sense if knowledge of the system dynamics is not used. The estimate may be used as a basis for further filtering and fusion techniques, such as sensor fusion with rate gyro data. The estimation algorithm is applied to the problem of state estimation for the Balancing Cube, a rigid structure that can actively balance on its corners. Experimental results are provided.**

I. Introduction

Attitude estimation refers to the problem of determining the rotation of a rigid body relative to an inertial frame of reference. It is a common problem in many engineering disciplines such as robotics, aeronautics, and space engineering. Accelerometers are used to estimate the pitch and roll components of attitude in many applications.[1] For the simple case of a non-moving rigid body, a single body-fixed tri-axis accelerometer is enough to determine tilt: the accelerometer measures the gravity vector in the body frame, which directly relates to the tilt angles pitch and roll of the rigid body. This is only true for a non-moving rigid body, however: if the body is rotated or accelerated, the body-fixed accelerometer also measures angular and centripetal acceleration terms. In other words, with a single tri-axis accelerometer on a moving body one cannot distinguish whether acceleration is due to motion or due to gravity.

One method of compensating for these dynamic effects is to combine accelerometers with rate gyros, which measure instantaneous angular velocity. In [1]–[3], rates gyros (partly in combination with other sensors) are used to determine the dynamics of a rigid body and correct the accelerometer's gravity vector observation. Another approach is to integrate the gyro measurements to obtain an estimate for attitude. This estimate is only accurate at high frequencies, however, as drift over time causes errors to accumulate. In complementary filtering, [4], [5], the gyrometric estimate is thus high-pass filtered and fused with a low-pass filtered accelerometer-based estimate. In [6], [7], estimators capable of switching between an accelerator-based estimate (for near static conditions) and a gyro-based estimate (for dynamic conditions) are used.

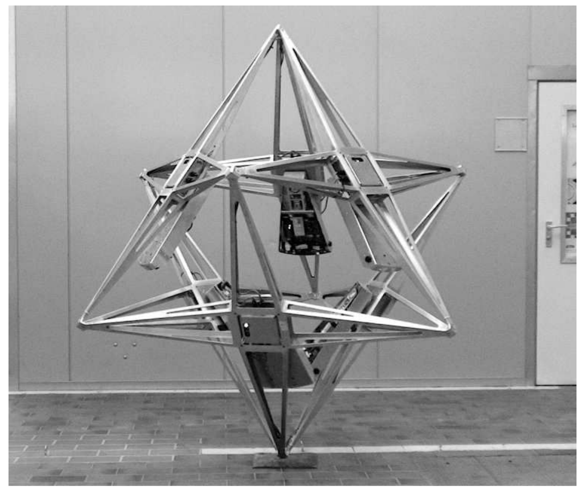

Fig. 1. The Balancing Cube, balancing on one of its corners.

In this work, the special case of tilt estimation for a rigid body with no translational degrees of freedom (but with full rotational freedom) is considered. The measurements from multiple accelerometers on the rigid body combined with the fact that the system rotates about a fixed point allows the derivation of a global tilt estimate that is independent of the rigid body dynamics. That is, the method works both for low and high frequency motion. The algorithm is easy to implement and requires only geometric information (knowledge of the sensor positions), but no dynamic model of the system. For the proposed tilt estimation method, four tri-axis accelerometers are needed. The tilt estimate, which is solely based on accelerometer data, may be used as a basis for further filtering techniques. A straight forward extension for fusing the accelerometic estimate with rate gyro data is presented in order to improve the noise characteristics of the estimate.

The developed algorithm is applied to estimate the pitch and roll angles of the *Balancing Cube* (Fig. 1), a 1.2 m cube that can balance on any of its corners.[2] The passive structure of the cube owes its ability to balance to six rotating

S. Trimpe and R. D'Andrea are with the Institute for Dynamic Systems and Control (IDSC), ETH Zurich, Sonneggstr. 3, 8092 Zurich, Switzerland {strimpe, rdandrea}@ethz.ch.

[1]The standard convention of yaw, pitch and roll angles is used to specify attitude. The yaw angle denotes the rotation about the gravity vector, and pitch and roll are successive rotations about the other two axes of the body frame, which will be made precise in Sec. II. Since pitch and roll together specify the orientation of a rigid body relative to the gravity vector, they are referred to as *tilt*.

[2]More information on the Balancing Cube may be found on the project website http://www.idsc.ethz.ch/Research_DAndrea/Cube. A short video of the cube accompanies this paper.

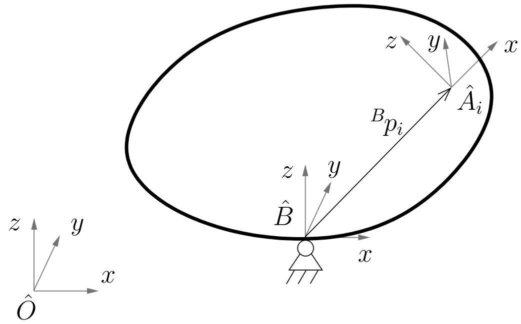

Fig. 2. The problem setup: a rigid body that has full rotational degrees of freedom. $\hat{O}$ denotes the inertial coordinate system, $\hat{B}$ is affixed to the rigid body with origin at the pivot, and $\hat{A}_i$ is the coordinate system of the ith sensor. The origins of $\hat{O}$ and $\hat{B}$ coincide (not shown).

mechanisms located on each inner face of the cube that move in concert to achieve equilibrium for the overall system. The tilt estimate of the cube is used as feedback information to stabilize the system. The Balancing Cube can be perceived as an example of a 3D pendulum. Pendulum systems are widely used for research in dynamics and control; for an overview, see [8] and references therein. The Balancing Cube is being used as a platform for research on distributed estimation and control and adaptation techniques.

This paper is organized as follows: The tilt estimation algorithm based on accelerometer measurements is derived in Sec. II. It is applied to the Balancing Cube in Sec. III and augmented by a straight-forward fusion algorithm with rate gyro data. Experimental results to demonstrate the proposed method are presented. Concluding remarks are given in Sec. IV.

II. Tilt Estimation Algorithm

In this section, an estimator is derived for the pitch and roll angles of a rigid body that has only rotational degrees of freedom. The estimate is based on measurements of multiple accelerometers mounted on the rigid body, whose mounting (location and orientation) is assumed to be known. The estimation problem is formally stated in Sec. II-A. In Sec. II-B, an estimate is derived for the gravity vector in the body frame, which is then used to determine pitch and roll angles in Sec. II-C.

A. Problem Formulation

We consider a rigid body that is supported by a fixed, frictionless pivot, see Fig. 2. Thus, the body has three rotational, but no translational degrees of freedom. The body-fixed coordinate frame $\hat{B}$ has its origin at the center of rotation. The inertial frame of reference is denoted by $\hat{O}$; the origin coincides with the origin of $\hat{B}$. On the body, there are L sensors (accelerometers) mounted at positions $p_i, i = 1, \dots, L$. The sensor positions are assumed to be known in the body frame of reference, ${}^B p_i$. Each sensor measures local accelerations along the three axes of its local frame $\hat{A}_i$.

Adopting notation from [9], two rotation matrices are introduced capturing the rotation of the rigid body and the mounting of the sensors:

- ${}^O_B R$ denotes the rotation of the inertial frame $\hat{O}$ to the body frame $\hat{B}$, and
- ${}^{A_i}_B R$ denotes the rotation of the local frame of the sensor i to the body frame.

Note that a vector quantity v given in frame $\hat{B}$ ${}^B v$, can be expressed in frame $\hat{O}$ by ${}^O v = {}^O_B R\, {}^B v$.

An accelerometer measures the acceleration ${}^O \ddot{p}_i$ at its mounting position ${}^O p_i$ plus the gravity vector ${}^O g$, rotated to its local frame $\hat{A}_i$:

$$ {}^{A_i} m_i = {}^{A_i}_B R\, {}^B_O R \left({}^O \ddot{p}_i + {}^O g \right) + {}^{A_i} n_i, \tag{1} $$

where ${}^{A_i} m_i \in \mathbb{R}^3$ is the ith accelerometer measurement and ${}^{A_i} n_i$ is measurement noise, which is assumed to be zero-mean, band-limited white noise with standard deviation σ_n, i.e. $\mathbb{E}\left[{}^A n_i\right] = 0$, $\mathbb{E}\left[{}^A n_i ({}^A n_i)^T\right] = \sigma_n^2 I_3$, where $\mathbb{E}\left[\cdot\right]$ denotes the expected value and I_3 is the identity matrix of dimension three by three. This noise model is reasonable for many MEMS based accelerometers once the bias has been removed.

From ${}^O p_i = {}^O_B R\, {}^B p_i$ and the fact that ${}^B p_i$ is constant with time, it follows for the acceleration of the point ${}^O p_i$,

$$ {}^O \ddot{p}_i = {}^O_B \ddot{R}\, {}^B p_i, \tag{2} $$

where ${}^O_B \ddot{R}$ denotes the second derivative of the rotation matrix ${}^O_B R$ with respect to time. The matrix ${}^O_B \ddot{R}$ captures the dynamic terms of the rigid body motion, i.e. rotational and centripetal acceleration terms. Using (2), the accelerometer measurement (1) can be rewritten as

$$ {}^{A_i} m_i = {}^{A_i}_B R\, {}^B_O R \left({}^O_B \ddot{R}\, {}^B p_i + {}^O g \right) + {}^{A_i} n_i. \tag{3} $$

Since all orientations of the sensors ${}^{A_i}_B R$ are assumed to be known, one can express all accelerometer measurements in body coordinates by multiplying (3) by ${}^B_{A_i} R = {}^{A_i}_B R^T$ from the left:

$$ {}^B m_i = \tilde{R}\, {}^B p_i + {}^B g + {}^B n_i, \tag{4} $$

where $\tilde{R} := {}^B_O R\, {}^O_B \ddot{R}$ combines the body rotation and the dynamic terms of the body motion,

$$ {}^B g = {}^B_O R\, {}^O g \tag{5} $$

is the gravity vector in body coordinates, and ${}^B n_i = {}^B_{A_i} R\, {}^{A_i} n_i$ is the noise vector rotated to the body frame. The mean and variance of the noise still satisfy $\mathbb{E}\left[{}^B n_i\right] = 0$ and $\mathbb{E}\left[{}^B n_i ({}^B n_i)^T\right] = \sigma_n^2 I_3$.

Assume that measurements are acquired at a rate T; introducing time index k, equation (4) can be rewritten,

$$ {}^B m_i(k) = \tilde{R}(k)\, {}^B p_i + {}^B g(k) + {}^B n_i(k). \tag{6} $$

Given the sensor measurements (6) for $i = 1, \dots, L$ at time k, the ultimate objective is to estimate the tilt of the rigid body (captured by ${}^B_O R$) at time k. As an intermediate step, an estimate is derived for the gravity vector ${}^B g(k)$ and (as a by-product) for the matrix $\tilde{R}(k)$ (Sec. II-B). The gravity vector estimate is then used to determine the tilt angles of the rigid body (Sec. II-C).

B. Optimal gravity vector estimation

In this section, the problem of estimating the gravity vector ${}^{B}g$ given the acceleration measurements (6), $i = 1, \ldots, L$ is formulated as a least-squares problem.

All L measurements of the form (6) are combined into a matrix equation (index k dropped for ease of notation),

$$\begin{aligned} M &= QP + N, && (7)\\ M &:= \begin{bmatrix} {}^{B}m_1 & {}^{B}m_2 & \ldots & {}^{B}m_L \end{bmatrix} \quad \in \mathbb{R}^{3\times L}, && (8)\\ Q &:= \begin{bmatrix} {}^{B}g & \tilde{R} \end{bmatrix} \quad \in \mathbb{R}^{3\times 4}, && (9)\\ P &:= \begin{bmatrix} 1 & 1 & \ldots & 1 \\ {}^{B}p_1 & {}^{B}p_2 & \ldots & {}^{B}p_L \end{bmatrix} \quad \in \mathbb{R}^{4\times L}, && (10)\\ N &:= \begin{bmatrix} {}^{B}n_1 & {}^{B}n_2 & \ldots & {}^{B}n_L \end{bmatrix} \quad \in \mathbb{R}^{3\times L}, \end{aligned}$$

where M combines all sensor measurements, Q is the unknown parameter matrix, P is the matrix of known parameters (sensor locations), and N combines all noise vectors, i.e. $\mathbb{E}[N] = 0$, $\mathbb{E}[N^T N] = 3\sigma_n^2 I_L = \sigma_N^2 I_L$, with $\sigma_N := \sqrt{3}\sigma_n$.

Besides the sought gravity vector ${}^{B}g$, the unknown matrix Q also contains the matrix $\tilde{R}$. In the following, a scheme is presented to optimally estimate the entire matrix Q, although the primary interest for tilt estimation is the gravity vector. In other applications, one might also be interested in an estimate of the dynamic terms ${}^{O}_{B}\ddot{R}$, which can be derived from $\tilde{R} = {}^{B}_{O}R\, {}^{O}_{B}\ddot{R}$ once ${}^{B}_{O}R$ is known.

The objective is to obtain an estimate $\hat{Q}^*$ of the matrix Q that minimizes

$$\min_{\hat{Q}} \mathbb{E}\left[\|\hat{Q} - Q\|_F^2\right] \quad \text{subj. to} \quad \mathbb{E}\left[\hat{Q}\right] = Q, \tag{11}$$

where $\|\cdot\|_F$ denotes the Frobenius matrix norm. The parameter estimate $\hat{Q}$ is restricted to linear combinations of the measurements M, i.e. the optimal X^* is sought for the factorization $\hat{Q} = MX$. This will yield a straightforward implementation: at each time step, the estimate $\hat{Q}$ is obtained by a single matrix multiplication.

Note that M, Q, and N in (7) are time-varying. At any time k, an optimal estimate for Q is sought given the set of measurements M.

The following lemma states the best unbiased linear estimate of the full parameter matrix Q:

Lemma 2.1 (Full Estimation Problem): Given the real matrices $P \in \mathbb{R}^{4\times L}$ and $M \in \mathbb{R}^{3\times L}$ satisfying $M = QP+N$ with unknown matrix $Q \in \mathbb{R}^{3\times 4}$ and the matrix random variable $N \in \mathbb{R}^{3\times L}$ with $\mathbb{E}[N] = 0$, $\mathbb{E}[N^T N] = \sigma_N^2 I_L$. Assuming P has full row rank, the (unique) minimizer $X^* \in \mathbb{R}^{L\times 4}$ of

$$\min_{X} \mathbb{E}\left[\|MX - Q\|_F^2\right] \quad \text{subj. to} \quad \mathbb{E}\left[MX\right] = Q \tag{12}$$

is given by

$$X^* = P^T (PP^T)^{-1}. \tag{13}$$

The minimum estimation error is

$$\mathbb{E}\left[\|MX^* - Q\|_F^2\right] = \sigma_N^2 \sum_{i=1}^{4} \frac{1}{s_i^2(P)}, \tag{14}$$

where $s_i(P)$ denotes the ith largest singular value of P.

Proof: Since $\mathbb{E}[MX] = \mathbb{E}[M]X = QPX$, it is required that

$$PX = I \tag{15}$$

to satisfy $\mathbb{E}[MX] = Q$. Next, consider the singular value decomposition (SVD) of P,

$$P = U \begin{bmatrix} \Sigma & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}, \tag{16}$$

with $U \in \mathbb{R}^{4\times 4}$ unitary, $\Sigma \in \mathbb{R}^{4\times 4}$ diagonal, $V_1 \in \mathbb{R}^{L\times 4}$, $V_2 \in \mathbb{R}^{L\times (L-4)}$, and $V = [V_1\, V_2]$ unitary. From the full row rank assumption on P, it follows that Σ is positive definite. Therefore, a parametrization of all X that satisfy (15) is given by

$$X = V_1 \Sigma^{-1} U^T + V_2 \bar{X}, \tag{17}$$

where $\bar{X} \in \mathbb{R}^{(L-4)\times 4}$ is a free parameter matrix. Thus, $\bar{X}$ needs to be chosen such that (12) is minimized: Using (7), (15) and basic properties of the trace operation, [10], yields

$$\begin{aligned} &\mathbb{E}\left[\|MX - Q\|_F^2\right] = \mathbb{E}\left[\|NX\|_F^2\right] \\ &= \mathbb{E}\left[\mathrm{trace}(X^T N^T N X)\right] = \mathrm{trace}\left(\mathbb{E}[N^T N] X X^T\right) \\ &= \sigma_N^2 \mathrm{trace}\left(XX^T\right) = \sigma_N^2 \mathrm{trace}\left(V^T X (V^T X)^T\right) \\ &= \sigma_N^2 \left\| \begin{bmatrix} \Sigma^{-1} U^T \\ \bar{X} \end{bmatrix} \right\|_F^2, \end{aligned} \tag{18}$$

which is minimized by $\bar{X} = 0$. Therefore,

$$X^* = V_1 \Sigma^{-1} U^T = P^T (PP^T)^{-1},$$

which can readily be seen by inserting (16) for P, and

$$\mathbb{E}\left[\|MX^* - Q\|_F^2\right] = \sigma_N^2 \|\Sigma^{-1} U^T\|_F^2 = \sigma_N^2 \|\Sigma^{-1}\|_F^2.$$

■

The optimal estimate $\hat{Q} = MX^*$ includes both the optimal estimate of the gravity vector ${}^{B}g$ and of the dynamics matrix $\tilde{R}$. Since, for tilt estimation, only the former is of interest, one needs to ask if X^* is also optimal if one seeks only an estimate of parts of the unknown matrix Q. The following lemma states that this is indeed the case.

Lemma 2.2 (Partitioned Estimation Problem): Let the matrices Q, P, N, M be defined as in Lemma 2.1. Furthermore, let $Q = [Q_1 \;\; Q_2]$, with $Q_1 \in \mathbb{R}^{3\times q}$, $Q_2 \in \mathbb{R}^{3\times (4-q)}$, $1 \le q \le 4$. Assuming P has full row rank, the (unique) minimizer $Y^* \in \mathbb{R}^{L\times q}$ of

$$\min_{Y} \mathbb{E}\left[\|MY - Q_1\|_F^2\right] \quad \text{subj. to} \quad \mathbb{E}\left[MY\right] = Q_1 \tag{19}$$

is $Y^* = X_1^*$, where $X^* = [X_1^*\, X_2^*]$ is the solution of Lemma 2.1.

Proof: It needs to be shown that $Y = X_1^*$ satisfies (19). First, since $X^* = [X_1^*\; X_2^*]$ satisfies $\mathbb{E}[MX] = Q$ in (12),

$$\begin{aligned} &\mathbb{E}\left[[MX_1^*\; MX_2^*]\right] = \mathbb{E}[MX^*] = Q = [Q_1\, Q_2] \\ &\Rightarrow \quad \mathbb{E}[MX_1^*] = Q_1 \quad (\text{and } \mathbb{E}[MX_2^*] = Q_2). \end{aligned}$$

Then,

$$\begin{aligned} \|MX^* - Q\|_F^2 &= \left\|\begin{bmatrix} MX_1^* - Q_1 & MX_2^* - Q_2 \end{bmatrix}\right\|_F^2 \\ &= \|MX_1^* - Q_1\|_F^2 + \|MX_2^* - Q_2\|_F^2, \end{aligned}$$

i.e. X^* minimizes both terms in the last expression separately and X_1^* thus minimizes $\|MX_1^* - Q_1\|_F^2$ alone. ∎

Applying Lemma 2.2 with $q = 1$ yields the optimal gravity vector estimate ${}^B\hat{g}(k)$ at time k given all sensor measurements $M(k)$,

$$ {}^B\hat{g}(k) = M(k)\, X_1^*, \tag{20} $$

with $X_1^* \in \mathbb{R}^{L\times 1}$. The *optimal fusion vector* X_1^* is static and completely defined by the geometry of the problem (through P) and can thus be computed offline.

Note that the gravity vector estimate (20) is independent of the rigid body dynamics, which are captured in ${}^O_B\ddot{R}$ (and thus in $\tilde{R}$). This can be seen from

$$ \begin{aligned} {}^B\hat{g} &= MX_1^* = QPX_1^* + NX_1^* \\ &= \begin{bmatrix} {}^Bg & \tilde{R} \end{bmatrix} \underbrace{U\Sigma V_1^T}_{P} \underbrace{V_1\Sigma^{-1}U_1^T}_{X_1^*} + NX_1^* \\ &= \begin{bmatrix} {}^Bg & \tilde{R} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} U_1^T + NX_1^* \\ &= \begin{bmatrix} {}^Bg & \tilde{R} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + NX_1^* = {}^Bg + NX_1^*, \end{aligned} $$

where the SVD of P (16) has been used. Clearly, the matrix $\tilde{R}$ does not appear in the estimate, i.e. the gravity vector observation is not corrupted by any dynamic terms. As expected, the sensor noise does enter the estimation equation.

Both in Lemma 2.1 and 2.2, the matrix P, which contains the sensor locations on the rigid body, is assumed to have full row rank. In the following, a physical interpretation of this rank condition is given.

Consider the case where P does not have full row rank. Then, there exists a nontrivial linear combination of the rows of P,

$$ \exists \lambda \neq 0 \in \mathbb{R}^4 :\ \lambda_1\, p_x + \lambda_2\, p_y + \lambda_3\, p_z + \lambda_4\, \mathbf{1} = 0, \tag{21} $$

where $p_x^T, p_y^T, p_z^T \in \mathbb{R}^{1\times L}$, denote the last three rows of P (the vectors of x, y, and z-coordinates of all sensor locations, respectively) and $\mathbf{1}^T \in \mathbb{R}^{1\times L}$, the vector of all ones, is the first row of P. Expression (21) is equivalent to

$$ \begin{aligned} &\exists \lambda \neq 0 \in \mathbb{R}^4 : \forall i = 1, \ldots, L, \\ &\lambda_1\, {}^Bp_{i,x} + \lambda_2\, {}^Bp_{i,y} + \lambda_3\, {}^Bp_{i,z} = -\lambda_4, \end{aligned} \tag{22} $$

where ${}^Bp_{i,x}, {}^Bp_{i,y}, {}^Bp_{i,z} \in \mathbb{R}$ denote the x, y, and z-coordinate of the ith sensor location in the body frame. Since the equation $\lambda_1 x + \lambda_2 y + \lambda_3 z = -\lambda_4$ defines a plane in (x, y, z)-space, condition (22) is equivalent to *all* L sensors lying on the same plane. Therefore, the full row rank condition on P is satisfied if and only if *not* all sensors lie on the same plane. Moreover, since three points always lie on a plane, this also implies that at least four tri-axis accelerometers are required for the proposed method.

Note that the gravity vector estimate given in Lemma 2.2 is optimal under the assumption that P has full row rank. These results can be extended and the rank condition on P can be relaxed when one seeks only the gravity vector. For example, one could directly measure gravity with a single tri-axis accelerometer at the pivot, where the dynamic terms do not enter the measurements. However, this is not possible for the Balancing Cube application.

C. Tilt estimation

With the estimate ${}^B\hat{g}$ of the gravity vector in the body frame, one can use (5) to estimate the body rotation, since the direction of the gravity vector in the inertial frame is known. In this work, the attitude of the rigid body is represented by z-y-x-Euler angles (yaw, pitch, roll), [9], i.e. the body frame $\hat{B}$ is obtained by rotating the inertial frame $\hat{O}$ successively about its z-axis, then the resulting y- and x-axis,

$$ {}^O_BR = R_z(\alpha)\, R_y(\beta)\, R_x(\gamma), \tag{23} $$

$$ R_z(\alpha) := \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix},\ R_y(\beta) := \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}, $$

$$ R_x(\gamma) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}, $$

where α, β, and γ are the yaw, pitch, and roll Euler angles, respectively. With this representation, the tilt of the rigid body is captured by β and γ. Using (23), (5) can be written as

$$ {}^Bg = {}^O_BR^T\ {}^Og = R_x^T(\gamma)\, R_y^T(\beta)\, R_z^T(\alpha)\ {}^Og. \tag{24} $$

Using ${}^Og = [0\ 0\ g_0]^T$ with gravity constant g_0 and the definitions of the rotation matrices, (24) simplifies to

$$ {}^Bg = R_x^T(\gamma)\, R_y^T(\beta)\ {}^Og = g_0 \begin{bmatrix} -\sin\beta \\ \sin\gamma\cos\beta \\ \cos\gamma\cos\beta \end{bmatrix}. \tag{25} $$

It follows that the z-Euler angle α is not observable from the accelerometer measurements.

Given the estimate of the gravity vector (20), the *accelerometric estimates* for the y- and x-Euler angles at time k are:

$$ \begin{aligned} \hat{\beta}_a(k) &= \text{atan2}\left(-{}^B\hat{g}_x(k), \sqrt{{}^B\hat{g}_y^2(k) + {}^B\hat{g}_z^2(k)}\right) \\ \hat{\gamma}_a(k) &= \text{atan2}\left({}^B\hat{g}_y(k), {}^B\hat{g}_z(k)\right), \end{aligned} \tag{26} $$

where atan2 is the four-quadrant inverse tangent. Note that one does not need to know the gravity constant g_0 for estimating tilt.

The variance of the angle estimates can be obtained from the variance of the gravity vector estimate, which can in turn be calculated from (18) and the SVD of P (16).

III. Application to the Balancing Cube

The proposed tilt estimation algorithm is applied to the Balancing Cube (Fig. 1). The passive structure of the cube is balanced on one of its corners by six rotating bodies (mounted on the inner faces of the cube) that shift their weight and exert forces on the structure, thereby keeping the system in equilibrium. The pitch and roll angle of the cube are estimated from measurements of six inertial measurement units (IMUs) with tri-axis accelerometers and rate gyros using the algorithm presented in Sec. II.

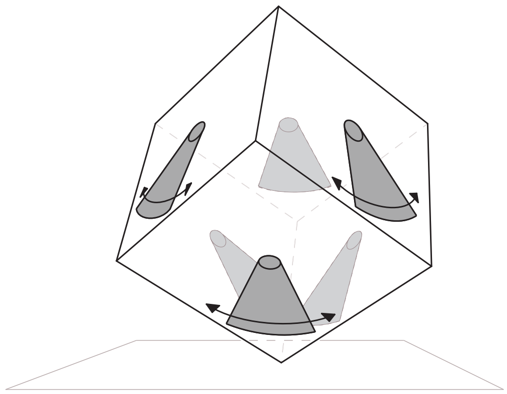

Fig. 3. Schematic drawing of the cube and the six rotating modules (gray).

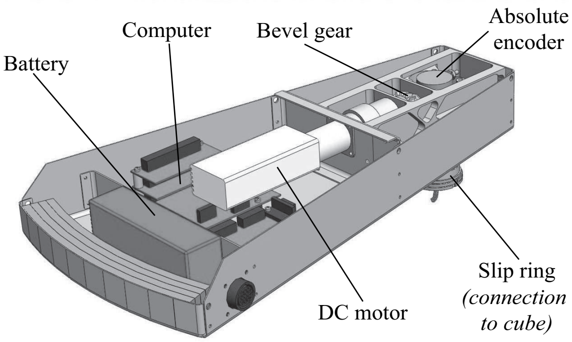

Fig. 4. CAD drawing of a module with its functional parts.

The Balancing Cube and the estimation problem are described in Sec. III-A. Furthermore, the design of the control architecture is explained. A comprehensive system description however is beyond the scope of this paper and will be published elsewhere. In Sec. III-B, the accelerometer-based estimator of Sec. II is applied to estimate the tilt of the cube. The noise level of the estimate is further reduced by straightforward fusion with data from the rate gyros. Experimental results are provided in Sec. III-C.

A. System description

The Balancing Cube consists of a rigid body in the shape of a cube and six rotating eccentric bodies (called *modules*) on the inner faces of the cube, see Fig. 3. Each of the six faces of the cube consists of an X-shaped aluminum structure (compare Fig. 1 and 3); the edge length is 1.2 m. The total (passive) mass of the cube is 21.4 kg; each module has an (active) mass of 3.7 kg. There is the possibility to add additional weights on the modules (up to 1.9 kg) to increase the control authority.

The objective is to balance the cube on one of its corners. In this configuration, the rigid body has three rotational and no translational degrees of freedom, since one can assume that the cube pivot does not slip due to friction combined with its large mass. Near the center of each face, an IMU [11] is mounted that measures accelerations and angular velocities each along three axes. Hence, the cube falls into the class of systems considered in Sec. II-A.

The modules are actuated by a DC motor and rotate relative to the cube structure. A drawing of a module with its functional elements is shown in Fig. 4. When the modules rotate, they exert reactional and gravitational forces (by shifting the center of mass) on the cube structure. An absolute encoder is used on each module to measure the angle of a module relative to its mounting. Furthermore, each module carries its own power unit and a computer that is connected to the sensors and the DC motor. The computers of all modules are connected to each other over a CAN bus network. In the operation mode that is considered in this work, each computer broadcasts all its local sensor measurements. In particular, each module has access to all six IMU measurements.

A state feedback controller is designed that stabilizes the unstable equilibrium of an upright standing cube and downward pointing modules. Estimates for the cube angles are obtained from the tilt estimation algorithm of Sec. II. In this work, only the three lower modules are used as actuators; the others are held fixed at the downward position. Although the details of the controller design are beyond the scope of this paper, a brief explanation of the control system follows.

For each individual module, an inner feedback loop is closed for the module velocity. The inner-loop controllers ensure that velocity commands are tracked at a faster rate than the natural dynamics of the cube. This way, one can neglect nonlinear effects such as friction and backlash in the actuation mechanism. A linear dynamical model about the equilibrium that takes into account the effect of the inner loops is obtained using the time-scale separation technique described in [12]. This model is then used to design an outer-loop stabilizing controller.

Since all modules are broadcasting their sensor measurements, each module has access to the same information. In particular, each module can generate estimates of all system states: the module angles are measured by encoders; estimates of the module angular velocities result from the time-scale separation technique, [12]; and estimates of the cube's pitch and roll angles and their rates are derived from the IMU measurements as presented in the next section. Note that for balancing, knowledge of yaw is not required. Hence, a centralized full-state feedback LQR controller can be designed for stabilizing the system. This controller is implemented on each module.

B. Tilt estimation

The coordinate frame definitions and the locations of the six IMUs on the cube are indicated in Fig. 5. The position vectors of the sensors are

$$
\begin{aligned}
{}^B p_1 &= \begin{bmatrix}0.55 & 0.64 & 0.06\end{bmatrix}^T, & {}^B p_2 &= \begin{bmatrix}0.56 & 0.06 & 0.65\end{bmatrix}^T, \\
{}^B p_3 &= \begin{bmatrix}0.06 & 0.55 & 0.64\end{bmatrix}^T, & {}^B p_4 &= \begin{bmatrix}0.64 & 0.55 & 1.14\end{bmatrix}^T, \\
{}^B p_5 &= \begin{bmatrix}0.56 & 1.14 & 0.55\end{bmatrix}^T, & {}^B p_6 &= \begin{bmatrix}1.14 & 0.55 & 0.56\end{bmatrix}^T.
\end{aligned}
$$

With this data, matrix P can be constructed as in (10). Applying Lemma 2.2 yields the optimal fusion vector for

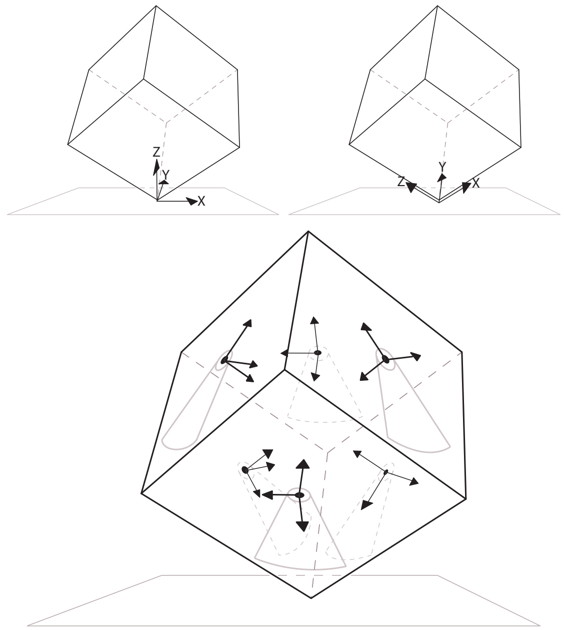

Fig. 5. Coordinate frames and sensor locations on the Balancing Cube: The inertial frame $\hat{O}$ is shown on the top, left figure; the body frame $\hat{B}$ is on the top, right figure; and the bottom shows the locations of the six IMUs and their local coordinate frames $\hat{A}_i$.

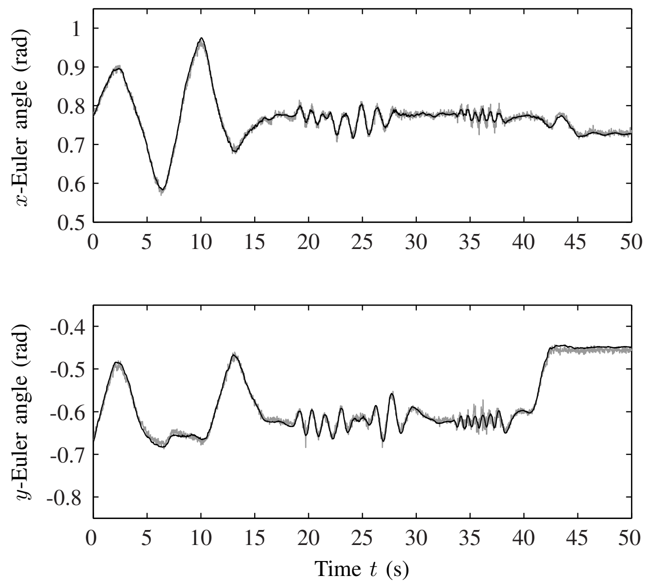

Fig. 6. Comparison of the accelerometric estimate of x- and y-Euler angles (gray) to camera-based reference measurement (black). The cube was moved manually around the nominal equilibrium at $\gamma_0 \approx 0.785$ and $\beta_0 \approx -0.615$.

estimating the gravity vector in the cube frame,

$$X_1^* = \begin{bmatrix} 0.78 & 0.78 & 0.68 & -0.52 & -0.42 & -0.30 \end{bmatrix}^T.$$

In the implementation of the algorithm, all accelerometer measurements are rotated to the body frame and stacked into the matrix $M(k)$ as in (8) at each time step. Then, (20) and (26) are implemented to obtain the accelerometric estimates for the pitch and roll angles, $\hat{\beta}_a(k)$ and $\hat{\gamma}_a(k)$.

In order to reduce the noise level of the accelerometer-based estimates, a straightforward scheme for data fusion with the tri-axis rate gyro measurements may be used. Let $r(k) \in \mathbb{R}^3$ denote the body angular rate at time k, which is directly measured by a gyro that is mounted on the body. Thus, an estimate $\hat{r}(k)$ of this quantity may be obtained by averaging the measurements of all six gyros. The body rates are transformed to Euler angular rates (see e.g. [13]) by

$$\begin{bmatrix} \hat{\dot{\alpha}}(k) \\ \hat{\dot{\beta}}(k) \\ \hat{\dot{\gamma}}(k) \end{bmatrix} = \begin{bmatrix} 0 & \sin\hat{\gamma}/\cos\hat{\beta} & \cos\hat{\gamma}/\cos\hat{\beta} \\ 0 & \cos\hat{\gamma} & -\sin\hat{\gamma} \\ 1 & \sin\hat{\gamma}\tan\hat{\beta} & \cos\hat{\gamma}\tan\hat{\beta} \end{bmatrix} \hat{r}(k), \tag{27}$$

which requires estimates of the Euler angles $\hat{\beta}$ and $\hat{\gamma}$. For a straightforward implementation the most recent estimate may be used as an approximation, i.e. $\hat{\beta} = \hat{\beta}(k-1)$ and $\hat{\gamma} = \hat{\gamma}(k-1)$.

Integrating the rate estimates (27) yields estimates for the Euler angles that are based on the rate gyro measurements. Thus, the accelerometer- and gyro-based estimates can be fused to obtain a better overall estimate of the cube pitch and roll angles,

$$\begin{aligned} \hat{\beta}(k) &= \kappa_1 \hat{\beta}_a(k) + (1-\kappa_1)\big(\hat{\beta}(k-1) + T\hat{\dot{\beta}}(k)\big) \\ \hat{\gamma}(k) &= \kappa_2 \hat{\gamma}_a(k) + (1-\kappa_2)\big(\hat{\gamma}(k-1) + T\hat{\dot{\gamma}}(k)\big), \end{aligned} \tag{28}$$

where T is the sampling time and κ_1 and κ_2 are tuning parameters that may be chosen such that the variance of the estimate is minimized given the noise specifications of accelerometers and rate gyros. For the application presented in this section, $\kappa_1 = \kappa_2 = 0.01$ was used.

C. Experimental results

The accelerometer-based tilt estimator (26) was implemented on the Balancing Cube as described in Sec. III-B. In order to verify the estimator, a global-positioning system was used that can track the position and attitude of rigid bodies at a rate of 200 frames per second based on infrared camera data. A description of this system may be found in [14]; a similar positioning system is also described in [15].

In the Balancing Cube application, the bias of the Euler angle estimates is corrected prior to operation in a calibration procedure, which accounts to first order for the accelerometer biases. For the comparison with the reference positioning system in this section, the DC component of the estimates is therefore not of interest. The means of the reference and estimator data presented in this section have thus been aligned for better comparability.

To demonstrate the accelerometer-based tilt estimator, the cube was put on a corner and moved by hand about the upright equilibrium position, which corresponds to the Euler angles $\gamma_0 = 45$ deg and $\beta_0 \approx -35$ deg. The results are shown in Fig. 6. Clearly, the accelerometric estimate is accurate both for slow and fast motion of the cube.

In contrast to the proposed method, in Fig. 7, the same experiment is shown, but now only a single tri-axis accelerometer (sensor $i = 4$) is used to observe the gravity vector. When the cube is static (from 45s to 50s), the estimate is accurate. However, when the cube is being moved, the estimate suffers from the dynamic terms that act as

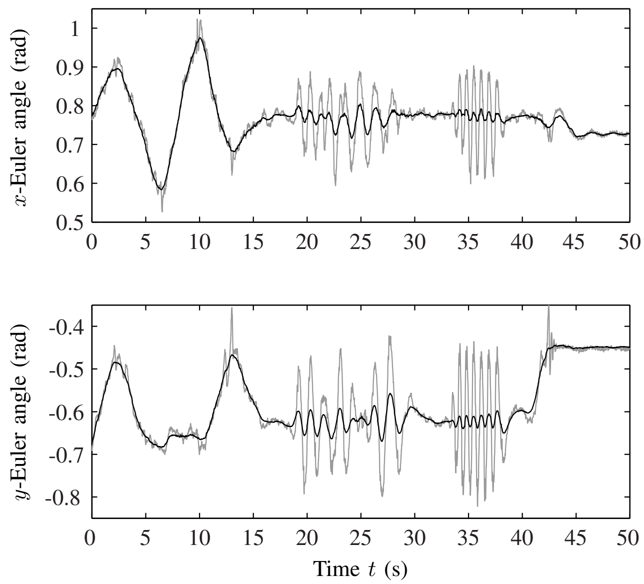

Fig. 7. Euler angle estimates if only a single tri-axis accelerometer is used (gray), compared to camera-based reference measurement (black).

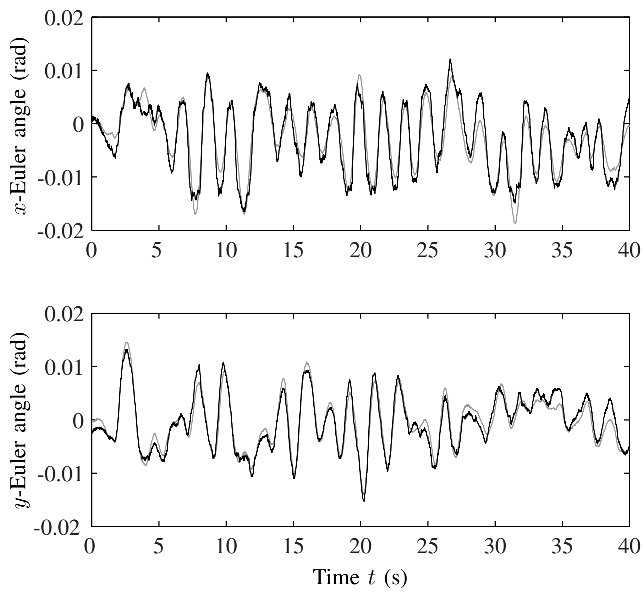

Fig. 8. Estimation data during balancing of the cube: accelerometric estimate fused with rate gyro data (gray) and camera-based reference measurements (black).

disturbances to the static estimator. This demonstrates that the dynamics are not negligible.

For the closed-loop operation of the system, i.e. for balancing the cube, the improved estimate for the cube angles (28) using both accelerometer and rate gyro data was used. The cube was balanced using only the three lower modules (with additional weights of 1.9 kg). The results are shown in Fig. 8 together with the camera-based reference data.

IV. CONCLUDING REMARKS

In comparison to existing methods, the main advantage of the presented tilt estimation technique is the independence of the estimate of any rigid body dynamics: the estimator is equally well applicable in quasi-static environments and in highly dynamic ones. This feature is mainly due to two facts: firstly, the fixed pivot of the rigid body and, secondly, the use of multiple accelerometers. Whether this technique can be extended to moving bodies, e.g. a body with a moving pivot, will be subject of future research.

The proposed method essentially transforms multiple accelerometer measurements into a tilt estimate. The tilt estimate is the optimal linear estimate in a least-squares sense if knowledge of the system dynamics is not used. Clearly, the accelerometer-based estimate may be used as the basis for further filtering techniques if knowledge of the system dynamics is available.

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